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Task-oriented maximally entangled states

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Abstract
We introduce the notion of a task-oriented maximally entangled state (TMES). This notion depends on the task for which a quantum state is used as the resource. TMESs are the states that can be used to carry out the task *maximally*. This concept may be more useful than that of a general maximally entangled state in the case of a multipartite system. We illustrate this idea by giving an operational definition of maximally entangled states on the basis of communication tasks of teleportation and superdense coding. We also give examples and a procedure to obtain such TMESs for n-qubit systems.

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1. Introduction

One of the greatest mysteries of quantum mechanics is the concept of entanglement. Ever since this idea was introduced by Schrödinger in 1935 [1], it has helped in unraveling the mystique of quantum mechanics. The entanglement of a quantum state of a system also allows us to carry out tasks, which would not be possible with a classical system. The entanglement properties of the quantum states are also responsible for the innate nonlocality [2] of the quantum mechanical framework. Although this idea is quite old, it is surprising that one still does not understand fully the entanglement properties of a tripartite system in a pure state, or even a bipartite system in a mixed state [3]. The proper characterization, quantification and classification of the entanglement properties of a multipartite system[1] is still a developing field. In this paper, we focus on systems in pure states.

A bipartite system in a pure state is the simplest system to examine and explore the entanglement properties. The nature of entanglement of such a system is well understood. Such a system has only one bipartite quantum correlation which can be characterized and quantified by the von Neumann entropy (or some other equivalent measure) [3, 4]. It is a suitable entanglement measure [5, 6] for such states. This measure maps a bipartite state to a real number. One can order the states according to von Neumann entropy.

[1] We categorize the systems into bipartite and multipartite. By a multipartite system we mean a system that consists of *more* than two subsystems.

Because of the existence of a suitable entanglement measure, there exists the concept of a maximally entangled state. This concept is well defined for bipartite systems which are in pure states and is independent of the entanglement measure. The measure von Neumann entropy is 0 for the unentangled systems and 1 for the maximally entangled systems. The states of the maximally entangled bipartite systems are Bell states which are given below in (1) and (2). Beyond a bipartite system, there is no consensus about what states might be considered maximally entangled. There are numerous suggestions [3] to characterize and quantify the entanglement features of a multipartite state. Some of these measures are quite difficult to compute and it is not clear if they satisfy the criteria to be a suitable measure [5, 6]. Without a suitable measure, what may be a maximally entangled state is not very clear. On the other hand, if one could identify the set of maximally entangled states, it could help in better understanding and classification of the multipartite entangled states.

In the literature, numerous attempts have been made to identify the maximally entangled state [7]. In this paper, we take a different approach to identify a maximally entangled state. In this approach, there may not exist an analog of Bell states for multipartite systems (indeed such a search may not be very useful). We suggest that for a multipartite system this notion may not be universal. There may exist global maxima relative to a specific entanglement measure, but the state with such a property may not be suitable for most tasks that one may envision. Therefore, we introduce the notion of a task-oriented maximally entangled state (TMES). TMESs are those states that may be suitable to carry out a specific task *maximally*. For different tasks, one may need different TMESs. A task can be Bell inequalities or some equalities which use the correlations of a quantum state or some communication or processing of information. Some of the communication-based tasks are teleportation [8], superdense coding [9], multi-receiver superdense coding [10], quantum cryptography [11], secret sharing [12], telecloning [13], etc. We present the concept of TMES by giving a definition of maximally entangled states on the basis of communication tasks of teleportation [8] and superdense coding [9]. A quantum state might be considered maximally entangled on the basis of resources available to these communication protocols. In this case, a TMES is the one which can help us to carry out the tasks of teleportation and superdense coding *maximally*. We can define maximality of these tasks as follows. For the maximal teleportation, an n-qubit state would allow us to teleport an unknown arbitrary $\frac{n}{2}$-qubit state when n is even and $\frac{(n-1)}{2}$-qubit state when n is odd. For maximal superdense coding, an n-qubit state would allow us to transmit n classical bits (cbits) of information by sending $\frac{n}{2}$ qubits when n is even and $\frac{(n+1)}{2}$ qubits when n is odd[2]. For example, when $n = 4$, the quadripartite GHZ state (given below in (3)) is not a TMES because one cannot teleport an unknown arbitrary two-qubit state. Furthermore, although one can transmit two cbits by sending one qubit and three cbits by sending two qubits, one cannot transmit four cbits by sending two qubits from Alice to Bob [10]. Therefore, this quadripartite state is not suitable for the maximal teleportation or maximal superdense coding.

It is not surprising that a suitable entanglement measure, e.g. von Neumann entropy, exists for bipartite states, since such states have only one bipartite correlation. However, in the case of a multipartite entangled state, there exist multiple bipartite, tripartite and higher correlations. Therefore, one number may not be suitable to characterize such states. One may need a set of numbers to characterize such states. Another way to look at this could be the following. All bipartite states can be characterized by just one parameter up to unary unitary

[2] There is an alternate way to define maximal superdense coding. Here the definition remains the same for even number of qubits. In the case of odd number of qubits, $n - 1$ cbits are transmitted by sending $\left(\frac{n-1}{2}\right)$ qubits. If we adopt this definition, then all the states that can do maximal teleportation can also do maximal superdense coding.

transformation equivalency. However, one needs more than one parameter to characterize a multipartite state. As an example, one needs six such real parameters to characterize a three-qubit state [14]. It would appear to be unlikely that the complexity and richness of the state of a multipartite system could be captured by just one number which would characterize the state as an entanglement measure. One may need a set of numbers, a vector measure, to capture the multifaceted entanglement properties of such states. Therefore, if we insist on finding an ordered list of multipartite entangled states, then it would depend on the correlations one is probing. In particular, there can be a number of different orderings which would depend on the task that we wish to perform using these states. Different tasks would depend on different quantum correlations which need to be maximal for that task. This naturally leads to the idea of TMESs.

2. TMESs for even number of qubits

For a two-qubit system, the TMESs are well known. These are Bell states:

$$|\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \tag{1}$$

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{2}$$

Using these states, one can carry out the conventional teleportation and superdense coding. One can teleport one-qubit state perfectly. One can also transmit two cbits by sending one qubit. Therefore, with Alice and Bob one qubit each, both protocols can be carried out *maximally*. Alice can convert these states into one another by applying unitary transformations $\{\sigma_0, \sigma_1, i\sigma_2, \sigma_3\}$ on her qubit. Here σ_0 is a 2×2 unit matrix, and $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices. As a point of interest, these Bell states can be obtained from the product states $\{|+\rangle|0\rangle, |-\rangle|0\rangle, |+\rangle|1\rangle, |-\rangle|1\rangle\}$ by applying the CNOT unitary operator U^{CN}. As this operator acts on two qubits, it can entangle them.

Let us now consider the case of larger number of qubits. Before considering a general n-qubit state, let us discuss the case of four-qubit states. There are a number of explicit states that have proposed as the maximally entangled state, as they seem to have some properties that may be desirable in a maximally entangled state. Some of these states are the GHZ state [15] , cluster state [16, 17], χ-state [18] and H-S state [19]:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \tag{3}$$

$$|\Omega\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle) \tag{4}$$

$$|\chi\rangle = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle) \tag{5}$$

$$|HS\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |1100\rangle + \omega(|1010\rangle + |0101\rangle) + \omega^2(|1001\rangle + |0110\rangle)). \tag{6}$$

Here $\omega = \mathrm{e}^{\frac{2\pi i}{3}}$. We would note that these are prototype states. By applying appropriate multi-unary[3] unitary transformations we can construct orthogonal states which would serve the same purpose. It is just like obtaining four Bell states from any one of the Bell states by applying appropriate unary unitary transformations. These multi-unary transformations, by

[3] The unary transformations act on only one qubit. The multi-unary transformations act separately on qubits of a multi-qubit system; they are direct product of unary transformations. The multinary transformations act on multiple qubits at the same time. These are entangled transformations. They can change the entanglement properties of the system.

definition, act on qubits separately. There are various arguments that have been put forward to support the case of each state. Some of these are as follows. The GHZ state has all one-qubits in a completely mixed state. The $|\Omega\rangle$ state shows certain features of entanglement which are persistent. The $|\chi\rangle$ state can be used for what we call maximal teleportation and superdense coding. The $|HS\rangle$ state has maximized two-qubit correlations. As we shall see later, these states can be TMESs, but for different tasks. But let us first construct TMESs for the tasks of teleportation and superdense coding.

In the case of an even number of qubits, say $2d$, one can teleport at most an arbitrary unknown d-qubit state. The easiest way to do it is to use d Bell states. Then each qubit of the d-qubit state can be teleported using one of the Bell states. This means that for a larger number of qubits, one could take a direct product of Bell states as the quantum resource. However, as these resource states are not genuinely multipartite entangled states, this resource is not very interesting. However, it turns out that by a multinary[4] unitary transformation, one could convert these unentangled states into entangled states. Since the teleportation protocol is not affected by these unitary transformations, these states with genuine multipartite entanglement also serve the purpose of teleporting a d-qubit state. We can see this as follows. Suppose we wish to teleport an arbitrary unknown n-qubit state $|\psi\rangle$ using the m-qubit state $|R\rangle$ as a quantum resource ($m \geqslant 2n$). If the teleportation is successful, then we would be able to write

$$|\psi\rangle_{a_1a_2\cdots\cdots a_n}|R\rangle_{b_1b_2\cdots\cdots b_m} = \frac{1}{2^n}\sum_{i=1}^{2^{2n}}|\mathcal{O}^i\rangle_{a_1a_2\cdots\cdots a_nb_1b_2\cdots\cdots b_{m-n}}{V^i}^{\dagger}_{b_{m-n+1}b_{m-n+2}\cdots\cdots b_m}|\psi\rangle_{b_{m-n+1}b_{m-n+2}\cdots\cdots b_m}. \tag{7}$$

Here subscripts are particle labels. $|\mathcal{O}^i\rangle$ are a set of orthogonal states and V^i are unitary operators. Alice would need to send $2n$ cbits of information to Bob to teleport an n-qubit state. If we apply a unitary transformation $I_{a_1a_2\cdots\cdots a_n}\otimes U_{b_1\cdots\cdots\cdots b_{m-n}}$ to both sides of the equation, then this transformation would convert the set of orthogonal states $|\mathcal{O}^i\rangle$ to another such set. So the teleportation would still be possible, but with a different quantum resource state, $U_{b_1\cdots\cdots\cdots b_{m-n}}|R\rangle_{b_1b_2\cdots\cdots b_m}$. Therefore, if we know a quantum resource state (e.g. a product of Bell states) that would allow the teleportation of an arbitrary unknown n-qubit state, then we can find another resource state by applying appropriate unitary transformation to it [21]. Alice can apply this unitary transformation. (Instead of Alice, Bob can also do it.) Interestingly, these entangled states can also be used for *maximal* superdense coding for an even number of qubits. This may not be surprising because of the close relationship between the two tasks. From the procedure, it is also clear that if a resource state can be used to teleport an n-qubit state, it can also be used to teleport any p-qubit state such that $p < n$. We illustrate the procedure for the four-qubit systems.

With a four-qubit entangled state, maximal teleportation will be that of an arbitrary and unknown two-qubit state:

$$|\psi\rangle_{ab} = \alpha|00\rangle_{ab} + \beta|01\rangle_{ab} + \gamma|10\rangle_{ab} + \delta|11\rangle_{ab}. \tag{8}$$

Here α, β, γ and δ are complex numbers. As noted above one could use the direct product of two Bell states as a resource. Since there are four Bell states, there are 16 possibilities. Now we could apply a binary unitary operator on two particles of these two states. For concreteness, let us take a situation where Alice and Bob share two copies of the $|\varphi^+\rangle$ states. One copy has particles 1 and 2, and the second copy has particles 3 and 4. Let Alice have particles 1 and 3, and Bob have particles 2 and 4. Here, particles 1 and 3 and particles 2 and 4 are not entangled. Alice can now apply the CNOT operation on her qubits. This will entangle her qubits and one

[4] See footnote 3.

would obtain a genuine quadripartite entangled state. This state is actually an example of the cluster state:

$$\begin{aligned} U_{13}^{CN}|\varphi^+\rangle_{12}|\varphi^+\rangle_{34} &= \tfrac{1}{2}U_{13}^{CN}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{1234} \\ &= \tfrac{1}{2}(|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle)_{1234}. \end{aligned} \tag{9}$$

This is a cluster state and one can explicitly check that one can indeed use it to teleport the two-qubit unknown state and also for maximal superdense coding [10].

Using a different unitary operator, one can also generate $|\chi\rangle$ states. The U^{CN} operator has the following representation in the computational basis:

$$U^{CN} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{10}$$

Replacing σ_1 by σ_3 in this matrix will give us the cluster state given in (4). The binary unitary operator that is needed to convert $|\varphi^+\rangle|\varphi^+\rangle$ to the $|\chi\rangle$ state is

$$U^{\chi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_3 & \sigma_1 \\ \mathrm{i}\sigma_2 & \sigma_0 \end{pmatrix}. \tag{11}$$

The state $|\Omega\rangle$ and the state $|\chi\rangle$ are TMESs from the point of view of the tasks of teleportation and superdense coding. The GHZ state and the HS state are *not* TMESs for these tasks, as we can check that these states are not suitable for maximal teleportation and superdense coding. The transformations that can convert two Bell states to these states are also not unitary; this is as it should be. The HS state can be a TMES for the task of telecloning. This protocol requires maximized and equal bipartite correlations [13]. The HS state is designed that way. For quantum secret sharing, $|GHZ\rangle$, $|\Omega\rangle$ and $|\chi\rangle$ states can be TMESs, but the $|HS\rangle$ state cannot.

There are many different sets of TMESs from the point of view of maximal teleportation and superdense coding protocols. We have already seen two such sets. One set is that of 16 orthogonal cluster states, and the other is that of 16 orthogonal χ-states. These sets are obtained from the original states by applying multi-unary unitary transformations. One can generate different sets by applying different binary unitary operators on the product of two Bell states. In fact one can construct 16 linearly independent binary unitary operators that act on two qubits. So we could have 16 independent such sets. One interesting set of 16 linearly independent operators including the CNOT unitary operator, U^{CN}, is

$$\Gamma_1 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_0 \end{pmatrix},$$

$$\Gamma_5 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \Gamma_7 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$

$$\Gamma_8 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \Gamma_9 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_1 & 0 \end{pmatrix}, \quad \Gamma_{10} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{pmatrix},$$

$$\Gamma_{11} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_3 & 0 \end{pmatrix}, \quad \Gamma_{12} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_0 & 0 \end{pmatrix}, \quad \Gamma_{13} = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_1 & 0 \end{pmatrix},$$

$$\Gamma_{14} = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \Gamma_{15} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_3 & 0 \end{pmatrix}, \quad \Gamma_{16} = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_0 & 0 \end{pmatrix}.$$

If we wish all the above matrices to be real, then we can replace σ_2 by $\mathrm{i}\sigma_2$. These matrices are the representation of the operators in the computational basis. Interestingly, all of these operators will generate entangled states from the product of two Bell states. Similarly, one can obtain other sets including CNOT-like operators controlled-Y and controlled-Z:

$$U_Y = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad U_Z = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{12}$$

One can obtain other sets by applying unitary transformation on the above matrices.

To obtain TMESs for the six-qubit systems and beyond, one can apply the CNOT unitary operator on two pairs of qubits. For example, if Alice has qubits 1, 3 and 5, while Bob has qubits 2, 4 and 6, then Alice can apply U_{13}^{CN} and U_{35}^{CN} to obtain a TMES. As noted above this TMES is not unique. One could apply any other set of binary or even a multinary unitary operator. By applying such linearly independent unitary operators, one can generate all sets.

This process can be extended to any $2d$-qubit state. In particular, one could apply CNOT operations, or any other binary unitary operations suggested above, on suitable $d-1$ pairs of qubits. Or one may choose to apply a multinary unitary operator on any other subsets or on all d-qubits. This procedure will give us genuinely entangled n-qubit states (n even) which are TMESs. One way to obtain an independent set of such multinary operators is to extend the procedure that we have used to obtain the set $\{\Gamma_a\}$ that act on two qubits using the set $\{\sigma_a\}$ that act on one qubit. This procedure allows us to construct the matrices for $(d+1)$-qubits from those that act on d-qubits. For $d = 1$ and 2, we have given these matrices above. Let the independent set of the unitary operators that act on d-qubits be $\{\gamma_a\}$, where $a = 1, 2, \ldots, 2^{2d}$. Then for the $(d+1)$-qubits, we can construct 2^{2d+2} linearly independent unitary matrices Σ_a as follows:

$$\Sigma_1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} \gamma_2 & 0 \\ 0 & \gamma_3 \end{pmatrix}, \ldots\ldots, \qquad \Sigma_{2^{2d}-1} = \begin{pmatrix} \gamma_{2^{2d}-1} & 0 \\ 0 & \gamma_{2^{2d}} \end{pmatrix},$$

$$\Sigma_{2^{2d}} = \begin{pmatrix} \gamma_{2^{2d}} & 0 \\ 0 & \gamma_1 \end{pmatrix}, \qquad \Sigma_{2^{2d}+1} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & -\gamma_2 \end{pmatrix}, \qquad \Sigma_{2^{2d}+2} = \begin{pmatrix} \gamma_2 & 0 \\ 0 & -\gamma_3 \end{pmatrix}, \ldots\ldots,$$

$$\Sigma_{2^{2d+1}-1} = \begin{pmatrix} \gamma_{2^{2d}-1} & 0 \\ 0 & -\gamma_{2^{2d}} \end{pmatrix}, \qquad \Sigma_{2^{2d+1}} = \begin{pmatrix} \gamma_{2^{2d}} & 0 \\ 0 & -\gamma_1 \end{pmatrix},$$

$$\Sigma_{2^{2d+1}+1} = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix}, \qquad \Sigma_{2^{2d+1}+2} = \begin{pmatrix} 0 & \gamma_2 \\ \gamma_3 & 0 \end{pmatrix}, \ldots\ldots,$$

$$\Sigma_{3\times 2^{2d}-1} = \begin{pmatrix} 0 & \gamma_{2^{2d}-1} \\ \gamma_{2^{2d}} & 0 \end{pmatrix}, \qquad \Sigma_{3\times 2^{2d}} = \begin{pmatrix} 0 & \gamma_{2^{2d}} \\ \gamma_1 & 0 \end{pmatrix},$$

$$\Sigma_{3\times 2^{2d}+1} = \begin{pmatrix} 0 & \gamma_1 \\ -\gamma_2 & 0 \end{pmatrix}, \qquad \Sigma_{3\times 2^{2d}+2} = \begin{pmatrix} 0 & \gamma_2 \\ -\gamma_3 & 0 \end{pmatrix}, \ldots\ldots,$$

$$\Sigma_{2^{2d+2}-1} = \begin{pmatrix} 0 & \gamma_{2^{2d}-1} \\ -\gamma_{2^{2d}} & 0 \end{pmatrix}, \qquad \Sigma_{2^{2d+2}} = \begin{pmatrix} 0 & \gamma_{2^{2d}} \\ -\gamma_1 & 0 \end{pmatrix}.$$

It is our conjecture that using this procedure, one can, in principle, generate all suitable states that can be a quantum resource in teleporting an arbitrary and unknown d-qubit state and for superdense coding, i.e., carrying out the tasks maximally.

3. TMESs for an odd number of qubits

For the systems with an odd number of qubits, the first non-trivial system is that of three qubits. None of the three-qubit states can be used to teleport an unknown arbitrary two-qubit state [22]. This is also clear from the fact that the construction of this state requires only one Bell state, as we shall see below. Furthermore since the Hilbert space of a three-qubit system is only eight dimensional, one cannot transmit four cbits by transmitting two qubits. At most three cbits could be transmitted. To obtain a TMES for $2d + 1$ qubits, one can start with the tensor product of d Bell states and a computational basis state. In the case of three-qubit system, e.g., we could have

$$\begin{aligned} U_{13}^{CN}|\varphi^+\rangle_{12}|0\rangle_3 &= \tfrac{1}{\sqrt{2}}U_{13}^{CN}(|000\rangle_{123} + |110\rangle_{123}) \\ &= \tfrac{1}{\sqrt{2}}(|000\rangle_{123} + |111\rangle_{123}). \end{aligned} \tag{13}$$

This is a GHZ state. For a three-qubit system it is a TMES for the tasks of superdense coding and teleportation. We note there exists a subclass of W-states [23] which can also be used for teleportation:

$$|W_n\rangle = \frac{1}{\sqrt{2+2n}}(|100\rangle + \sqrt{n}|010\rangle + \sqrt{n+1}|001\rangle). \tag{14}$$

Since this state can be used for the maximal teleportation, one should be able to obtain it after applying suitable multinary unitary transformations on $|\varphi^+\rangle|0\rangle$. For example this transformation for the $|W_2\rangle$ state is

$$U^{W_2} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{15}$$

This state is also suitable for transmitting three cbits by sending two qubits, but not for all $2-1$ partitions of three qubits. Since our definition of the maximal superdense coding does not have any requirement for partitions, this state is also a TMES. We note that the product state $|\varphi^+\rangle|0\rangle$ can be used for maximal superdense coding also.

Let us now consider a system of five qubits. For such a system TMES would not be a GHZ state [10]. However, we can obtain TMESs by applying appropriate unitary transformations on the product state of two Bell states and a computational basis state. One way is

$$\begin{aligned} U_{13}^{CN}U_{35}^{CN}|\varphi^+\rangle_{12}|\varphi^+\rangle_{34}|0\rangle_5 &= \tfrac{1}{2}U_{13}^{CN}U_{35}^{CN}(|00000\rangle + |00110\rangle + |11000\rangle + |11110\rangle)_{12345} \\ &= U_{35}^{CN}\tfrac{1}{2}(|00000\rangle + |00110\rangle + |11100\rangle + |11010\rangle)_{12345} \\ &= \tfrac{1}{2}(|00000\rangle + |00111\rangle + |11101\rangle + |11010\rangle)_{12345}. \end{aligned} \tag{16}$$

This state is a TMES for a five-qubit system corresponding to the tasks of teleportation and superdense coding. This is because it can be used for maximal superdense coding and teleportation. One can teleport an unknown arbitrary two-qubit state and transmit five cbits by sending three qubits. This is also an example of cluster state for five qubits. As discussed in the case of an even number of qubit system, this state is not unique. By applying multi-unary unitary transformations on three qubits, one can obtain 32 orthogonal states, which will serve the same purpose. Furthermore, by applying linearly independent multinary unitary transformations, given in the last section, one can obtain linearly independent sets of TMESs. By maximizing negativity numerically, Brown *et al* [20] have obtained a 'highly entangled' five-qubit state. Their state is a TMES for our tasks. One can obtain it by applying an appropriate multinary unitary transformation. This state does not have all maximized bipartite correlations.

One can generalize this procedure to *any* odd number of qubit system. So for a $2d + 1$ qubit system, one has to consider the product state of d Bell states and a one-qubit state and apply a suitable multinary unitary transformation. This will make the $2d + 1$ qubits genuinely entangled. This can serve as the resource for maximal teleportation and superdense coding. In the last section we have given a more detailed procedure and sets of unitary matrices that can be used for the transformations.

4. Conclusions

We have argued that the universal maximally entangled state in the case of multipartite systems may not be generally useful, even if such a state could be identified. Instead the notion of a TMES may be more fruitful. A TMES for a given task is the state that can carry out the task *maximally*. This makes the notion of maximally entangled states task dependent. For different tasks, different states may allow us to carry out the task maximally. We have illustrated this idea on the basis of the communication tasks of teleportation and superdense coding. To this end, we have given a strategy to obtain instances of the TMESs corresponding to these tasks. We conjecture that using our procedure one would be able to obtain all states that may be suitable for the maximal teleportation and superdense coding. Similarly, for other tasks, one may obtain the suitable TMESs.

References

[1] Schrodinger E 1935 *Naturwissenchaften* **33** 807
[2] Bell J S 1964 *Physics* **1** 195
[3] Plenio M B and Virmani S 2005 arXiv:quant-ph/0504163 and references therein
[4] Popescu S and Rohrlich D 1997 *Phys. Rev.* A **56** R3319
[5] Vedral V and Plenio M B 1998 *Phys. Rev.* A **57** 1619
[6] Donald M J, Horodecki M and Rudolph O 2002 *J. Math. Phys.* **43** 4252
[7] Scott A J 2004 *Phys. Rev.* A **69** 052330
Facchi P, Florio G, Parisi G and Pascazio S 2008 *Phys. Rev.* A **77** 060304
[8] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wooters W K 1993 *Phys. Rev. Lett.* **70** 1895
[9] Bennett C H and Wiesner S 1992 *Phys. Rev. Lett.* **69** 2881
[10] Pradhan B, Agrawal P and Pati A K 2007 arXiv:quan-ph/0705.191
[11] Bennett C H and Brassard G 1984 *Proc. IEEE Int. Conf. on Computers, Systems, and Signal Processing (Bangalore, India)* (New York: IEEE)
Ekert A K 1991 *Phys. Rev. Lett.* **67** 661
[12] Hillery M, Buzek V and Berthiaume A 1999 *Phys. Rev.* A **59** 1829
[13] Murao M, Jonathan D, Plenio M B and Vedral V 1999 *Phys. Rev.* A **59** 156
[14] Eisert J and Gross D 2005 arXiv:quant-ph/0505149
[15] Greenberger D, Horne M and Zeilinger A 1989 *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* ed M Kaftos (Dordrecht: Kluwer)
[16] Briegel H J and Raussendorf R 2001 *Phys. Rev. Lett.* **86** 910
[17] Raussendorf R and Briegel H J 2001 *Phys. Rev. Lett.* **86** 5188
[18] Yeo Y and Chua W K 2006 *Phy. Rev. Lett.* **96** 060502
[19] Higuchi A and Sudbery A 2000 *Phys. Lett.* A **273** 213
[20] Brown I D K, Stepney S, Sudbery A and Braunstein S L 2005 *J. Phys. A: Math. Gen.* **38** 1119
[21] Chen P-X, Zhu S-Y and Guo G-C 2006 *Phys. Rev.* A **74** 032324
[22] Agrawal P and Pradhan B 2010 (in preparation)
[23] Agrawal P and Pati A 2006 *Phys. Rev.* A **74** 062320